SUPPLEMENTARY

Proof of Axioms

Proof. PROOF OF AXIOM 1
Start from the definition of $g(\mathcal{S})$,

$$g(\mathcal{S}) - g(\tilde{\mathcal{S}}) = \frac{1}{|\mathcal{S}|}\sum_{v_i \in \mathcal{M}} (1+\lambda) f_i(\mathcal{S}) - \lambda q_i(\mathcal{S}) - \frac{1}{|\tilde{\mathcal{S}}|}\sum_{v_i \in \mathcal{M}} (1+\lambda) f_i(\tilde{\mathcal{S}}) - \lambda q_i(\tilde{\mathcal{S}}).$$

Since $|\mathcal{S}|= |\tilde{\mathcal{S}}| \wedge \boldsymbol{f}(\tilde{\mathcal{S}}) > \boldsymbol{f}(\mathcal{S}) \wedge \boldsymbol{q}(\mathcal{S}) - \boldsymbol{f}(\mathcal{S}) = \boldsymbol{q}(\tilde{\mathcal{S}}) - \boldsymbol{f}(\tilde{\mathcal{S}})$, it is obviously that $g(\tilde{\mathcal{S}}) > g(\mathcal{S})$. ■

Proof. PROOF OF AXIOM 2
$|\mathcal{S}|= |\tilde{\mathcal{S}}| \wedge \boldsymbol{f}(\mathcal{S}) = \boldsymbol{f}(\tilde{\mathcal{S}})$, hence we have $\boldsymbol{q}(\tilde{\mathcal{S}}) - \boldsymbol{f}(\tilde{\mathcal{S}}) < \boldsymbol{q}(\mathcal{S}) - \boldsymbol{f}(\mathcal{S})$, for given $|\mathcal{S}|= |\tilde{\mathcal{S}}|$. Since $\boldsymbol{f}(\mathcal{S}) = \boldsymbol{f}(\tilde{\mathcal{S}})$, then $g(\tilde{\mathcal{S}}) > g(\mathcal{S})$. ■

Proof. PROOF OF AXIOM 3
Given the conditions, we have $\boldsymbol{f}(\mathcal{S}) = \boldsymbol{f}(\tilde{\mathcal{S}}) \wedge \boldsymbol{q}(\tilde{\mathcal{S}}) - \boldsymbol{f}(\tilde{\mathcal{S}}) = \boldsymbol{q}(\mathcal{S}) - \boldsymbol{f}(\mathcal{S})$. And since $g(\mathcal{S}) > 0$, $g(\tilde{\mathcal{S}}) > 0$, then $g(\tilde{\mathcal{S}})/g(\mathcal{S}) = |\mathcal{S}|/|\tilde{\mathcal{S}}|$. Since $|\tilde{\mathcal{S}}|< |\mathcal{S}|$, hence $g(\tilde{\mathcal{S}}) > g(\mathcal{S})$. ■

Proof of Lemmas

Proof. PROOF OF LEMMA 1
We can express $g(\mathcal{S})$ as follows:

$$g(\mathcal{S}) = \frac{1}{|\mathcal{S}|}\sum_{v_j \in \mathcal{M}} (\lambda+1) w_j(\mathcal{S})$$

Now, suppose $\exists v_i \in \mathcal{S}^*$ such that $(\lambda+1)w_i(\mathcal{S}^*) < g(\mathcal{S}^*)$, we have two cases to consider:
If $v_i \in \mathcal{M}^*$, by removing v_i from $\mathcal{S}^*$, we have:

$$g(\mathcal{S}^* \backslash v_i) = \frac{\sum_{v_j \in \mathcal{M}^*} (\lambda+1) w_j(\mathcal{S}^*) - (\lambda+1) w_i(\mathcal{S}^*)}{|\mathcal{S}^*|-1}$$

.
If $v_i \in \mathcal{A}^* \cup \mathcal{C}^*$, by removing v_i from $\mathcal{S}^*$, the out-degree or in-degree of the nodes connected to v_i in $\mathcal{M}^*$ is reduced, hence the minimum $f_j(\mathcal{S}^*)$ or maximum $q_j(\mathcal{S}^*)$ for $v_j \in \mathcal{M}^*$ is reduced. To obtain a lower bound of $g(\mathcal{S}^* \backslash v_i)$, we only need to consider the worst case that all the edges of v_i contributes to $f_j(\mathcal{S}^*)$ of the middle nodes they are connected to. Therefore, in the worst case, the numerator of objective is reduced at most $(1+\lambda)d_i(\mathcal{S}^*)$.
In both cases,

$$\begin{aligned} g(\mathcal{S}^* \backslash v_i) &\geq \frac{\sum_{v_j \in \mathcal{M}^*} (\lambda+1) w_j(\mathcal{S}^*) - (\lambda+1) w_i(\mathcal{S}^*)}{|\mathcal{S}^*|-1} \\ &> \frac{\sum_{v_j \in \mathcal{M}^*} (\lambda+1) w_j(\mathcal{S}^*) - g(\mathcal{S}^*)}{|\mathcal{S}^*|-1} \\ &= \frac{\sum_{v_j \in \mathcal{M}^*} (\lambda+1) w_j(\mathcal{S}^*)(1-1/|\mathcal{S}^*|)}{|\mathcal{S}^*|-1} \\ &= g(\mathcal{S}^*), \end{aligned}$$

which is a contradiction, since $g(\mathcal{S}^*)$ is the optimal solution. ■

Proof. PROOF OF LEMMA 2
There are two cases:
Case 1:$\forall v_i \in \mathcal{A}^* \cup \mathcal{C}^*$, we have $w_i(\mathcal{S}') = d_i(\mathcal{S}')$ and $w_i(\mathcal{S}^*) = d_i(\mathcal{S}^*)$. Since $\mathcal{S}^* \subseteq \mathcal{S}'$, then $d_i(\mathcal{S}') \geq d_i(\mathcal{S}^*)$. Then we have $w_i(\mathcal{S}') \geq w_i(\mathcal{S}^*)$.
Case 2:$\forall v_i \in \mathcal{M}^*$, $w_i(\mathcal{S}') - w_i(\mathcal{S}^*) = f_i(\mathcal{S}') - \frac{\lambda}{\lambda+1} q_i(\mathcal{S}') - (f_i(\mathcal{S}^*) - \frac{\lambda}{\lambda+1} q_i(\mathcal{S}^*))$. Since $S^* \subseteq S'$, then $d_i^+(S') \geq d_i^+(S^*)$ and $d_i^-(S') \geq d_i^-(S^*)$. Hence $f_i(S') >= f_i(S^*)$. Then we have $w_i(\mathcal{S}') - w_i(\mathcal{S}^*) \geq -\frac{\lambda}{\lambda+1}(q_i(\mathcal{S}') - q_i(\mathcal{S}^*))$, and $w_i(\mathcal{S}') \geq w_i(\mathcal{S}^*) - \frac{\lambda}{\lambda+1}(q_i(\mathcal{S}') - q_i(\mathcal{S}^*))$. ■

Proof of Complexity

Proof. PROOF OF COMPLEXITY
The most time consuming steps are in Lines 5 to 10 in in Alg 1 . The priorities of elements in $\mathcal{T}$ will change when removing a node v from subset $\mathcal{S}$. The changed nodes are the neighbors of v. At the end of iteration, almost all the nodes are removed once. Thus, overall, the algorithm updates the priority tree $O(k|\mathcal{E}|)$ times. Each update for tree $\mathcal{T}$ with size $|\mathcal{V}|$ requires $O(\log|\mathcal{V}|)$. Therefore, the time complexity of FlowScope is $O(k|\mathcal{E}|\log|\mathcal{V}|)$. ■

Proof of Theorems

Proof. PROOF OF IMPLICATION THEOREM 1
If no nodes in $\mathcal{S}^*$ are removed in the algorithm before all the other nodes are removed, then $\hat{\mathcal{S}} = \mathcal{S}' = \mathcal{S}^*$ and satisfies the bound.
If $v_i \in \mathcal{S}'$ contains other nodes than $\mathcal{S}^*$, then $\mathcal{S}^* \subset \mathcal{S}'$: since v_i is the next node removed in $\mathcal{S}'$ by FlowScope, then we have

$$g(\mathcal{S}') = \frac{1}{|\mathcal{S}'|}\sum_{v_j \in \mathcal{M}'} (1+\lambda) w_j(\mathcal{S}') \geq \frac{|M'|}{|S'|}(1+\lambda) w_i(\mathcal{S}')$$

And we then have two cases:
Case 1: If $v_i \in \mathcal{A}^* \cup \mathcal{C}^*$*:*
By the first part of Lemma 2 followed by Lemma 1, we have:

$$\frac{|M'|}{|S'|}(1+\lambda) w_i(\mathcal{S}') \geq \frac{|M'|}{|S'|}(1+\lambda) w_i(\mathcal{S}^*) \geq \frac{|M'|}{|S'|} g(S^*)$$

Case 2: If $v_i \in \mathcal{M}^*$*:*
Since $\mathcal{S}' \subseteq V$ so $q_i(\mathcal{S}') \leq q_i(\mathcal{V})$. Hence by the second part of Lemma 2 followed by Lemma 1, we have:

$$\begin{aligned} g(\mathcal{S}') &\geq \frac{|M'|}{|S'|}((1+\lambda) w_i(S^*) - \lambda(q_i(\mathcal{V}) - q_i(\mathcal{S}^*))) \\ &\geq \frac{|M'|}{|S'|}(g(S^*) - \lambda(q_i(\mathcal{V}) - q_i(\mathcal{S}^*))) \\ &\geq \frac{|M'|}{|S'|}(g(S^*) - \lambda\varepsilon) \end{aligned}$$

Moreover, $g(\hat{\mathcal{S}})$ returned by FlowScope is the maximum objective during the iterations, i.e. $g(\hat{\mathcal{S}}) \geq g(\mathcal{S}')$. Therefore our bound holds for both of the two cases. ■

Proof. PROOF OF IMPLICATION THEOREM 2
It can be inferred from Theorem 1 that

$$g(\mathcal{S}^*) \leq \frac{|\mathcal{S}'|}{|\mathcal{M}'|} g(\hat{\mathcal{S}}) + \lambda\varepsilon.$$

then,

$$\begin{aligned} n_0(\frac{|\mathcal{S}'|}{|\mathcal{M}'|} g(\hat{\mathcal{S}}) + \lambda\varepsilon) &\geq n_0 g(\mathcal{S}^*) \\ &= \sum_{v_i \in \mathcal{S}^*} f_i(\mathcal{S}^*) - \lambda(q_i(\mathcal{S}^*) - f_i(\mathcal{S}^*)) \\ &= \sum_{v_i \in \mathcal{S}^*} f_i(\mathcal{S}^*) - \lambda\eta \sum_{v_i \in \mathcal{S}^*} f_i(\mathcal{S}^*) \\ &= (1 - \lambda\eta) \sum_{v_i \in \mathcal{S}^*} f_i(\mathcal{S}^*) \end{aligned}$$

which implies

$$\frac{\sum_{v_i \in \mathcal{S}^*} f_i(\mathcal{S}^*)}{n_0} \leq \frac{1}{1 - \lambda\eta}(\frac{|S'|}{|M'|} g(\hat{S}) + \lambda\varepsilon)$$

■